THE INVESTIGATION OF THE BEHAVIOR OF LINEAR DYNAMICAL SYSTEMS UNDER THE ACTION OF NONLINEAR FUNCTIONS OF RANDOM PROCESSES

(ISSLEDOVANIIA POVEDENIIA LINEINYKH DINAMICHESKIKH SISTEM POD VOZDEISTVIEM NELINEINYKH FUNKTSII SLUCHAINYKH PROTSESSOV)

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A.A. SVESHNIKOV
(Leningrad)

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1. Statement of the problem. Much work has been done on the use of approximate methods of linearization in the analysis of nonlinear dynamical systems subjected to the action of random disturbances. In some such cases there arises the necessity for an exact solution of the problem, for the determination of the error in the approximate methods, or for the solution when the problem does not lend itself to the application of the method of linearization.

In this work there is considered the behavior of linear dynamical systems whose input is a nonlinear function of a stationary random process $X(t)$, i.e. a system described by the equation

$$\frac{d^n Y(t)}{dt^n} + a_1(t)\frac{d^{n-1} Y(t)}{dt^{n-1}} + \cdots + a_n(t) Y(t) = nf_j[X(t)] \tag{1.1}$$

The probability characteristics of the process $X(t)$ is assumed to be known. $Y(t)$ is the unknown function to be found, characterizing the state of the system, $a_l(t)$, $l = 1, \ldots, n$ are given functions of time. The nonlinear function $f_j(X)$ $(j = 1, 2, 3)$ will be assumed here as one of the following types:

$$f_1(X) = \operatorname{sign} X \tag{1.2}$$

$$f_2(X) = \tfrac{1}{2}[\operatorname{sign}(X - a) + \operatorname{sign}(X + a)] \tag{1.3}$$

$$f_3(X) = \tfrac{1}{2}[(X + a)\operatorname{sign}(X + a) - (X - a)\operatorname{sign}(X - a)] \tag{1.4}$$

The first function corresponds to the nonlinear link of the type "yes-no"; the second one to the link of the type "yes-no", but it has a zone of insensitivity; the third function corresponds to a link which has a linear interval and an interval of "saturation". The presence of a nonlinear link in the operation of a linear dynamical system makes the entire problem a nonlinear one, which complicates its solution, especially from the viewpoint of the theory of random functions.

The solution of Equation (1.1) (we assume for the sake of simplicity that we have zero initial conditions) can be written in the form

$$Y(t) = \int_0^t p(t, t_1) f_j [X(t_1)] \, dt_1 \tag{1.5}$$

Here $p(t, t_1)$ is the weight function of the system expressible in terms of a system of independent integrals (solutions) of the homogeneous equation corresponding to (1.1). Raising both parts of (1.5) to the degree m and applying the operation for finding the mathematical expectation to both sides of the obtained equation, one can easily establish that any moment of the ordinate of the random function $Y(t)$ can be expressed by means of an integral of an expression containing mixed moments of the random function $f_j[X(t)]$. For example, for the mathematical expectation and dispersion of the random quantity $Y(t)$ we obtain

$$M[Y(t)] = y(t) = \int_0^t p(t, t_1) M\{f_j [X(t_1)]\} \, dt_1 \tag{1.6}$$

$$D[Y(t)] = \sigma_y^2 = \int_0^t \int_0^t p(t, t_1) p(t, t_2) M\{f_j [X(t_1)] f_j [X(t_2)]\} \, dt_1 dt_2 - y^2(t)$$

In consequence of the hypothesis made at the very beginning on the stationary nature of the random function $X(t)$, the mathematical expectation $M\{f_j[X(t)]\}$ is constant, and $M\{f_j[X(t_1)]\, f_j[X(t_2)]\}$ will be a function of $\tau = t_2 - t_1$ and will not be dependent on the arguments t_1 and t_2 separately. Therefore, Formula (1.6) can be represented in the form

$$y(t) = M\{f_j(X(t)]\} \int_0^t p(t, t_1) \, dt_1 \tag{1.7}$$

$$\sigma_y^2 = \int_0^t \left\{ \int_\tau^{2t-\tau} p(t, \xi - \tau) p(t, \xi + \tau) \, d\xi \right\} M\{f_j [X(t)] f_j [X(t+\tau)]\} \, d\tau \tag{1.8}$$

Thus, the first two moments of the solution of Equation (1.1) can be found if one knows the mathematical expectations

$$\mu_j = M\{f_j [X(t)]\}, \qquad \nu_j(\tau) = M\{f_j [X(t)] f_j [X(t+\tau)]\} \tag{1.9}$$

If the first and second laws of the distribution of the random function $X(t)$ are known, then the determination of μ_j and ν_j can be carried out by means of the general formulas for finding the mathematical expectation, but the computations required for this are quite cumbersome even for the simplest law of distribution of the random function $X(t)$. It will be more advantageous to use a different method of evaluation in which the function $f_j(X)$ is represented as a Fourier transform of the transfer function corresponding to $f_j(X)$. This makes it possible to express the mathematical expectation in (1.9) as a characteristic function of the ordinate $X(t)$. For the application of this method it is, however, necessary either to assume the existence of the Fourier integral of the function $f_j(X)$ (which is not valid, for example, for the nonlinear types (1.2), (1.3) and (1.4)) or one has to select the appropriate form of the contour of integration [1].

These difficulties can be avoided if one uses the integral representation of the right-hand side of (1.1) by the method which was applied by Markov [2] for the proof of the limit theorem of the theory of probability, and which is based on the use of the Dirichlet integral. In accordance with the latter we have

$$\operatorname{sign} x = \frac{1}{\pi i}\int_{-\infty}^{\infty} e^{iux}\frac{du}{u} \tag{1.10}$$

Formula (1.7) makes it possible to obtain an integral representation for $f_j(X)$ whose substitution into (1.9) yields at once the moments of the function $Y(f)$ in terms of the characteristic function of the ordinates $X(t)$.

2. Evaluation of the moments of an essentially nonlinear expression of the type $f_j[X(t)]$. In view of (1.2) and (1.7) we have

$$f_1[X(t)] = \frac{1}{\pi i}\int_{-\infty}^{\infty} e^{iuX(t)}\frac{du}{u} \tag{2.1}$$

Hence

$$\mu_1 = \frac{1}{\pi i}\int_{-\infty}^{\infty} M[e^{iuX(t)}]\frac{du}{u} = \frac{1}{\pi i}\int_{-\infty}^{\infty} E(u)\frac{du}{u} \tag{2.2}$$

In an analogous manner, we obtain for the mixed moments of the random quantities $f_j[X(t_1)]$ and $f_j[X(t_2)]$ the expression

$$\nu_1 = -\frac{1}{\pi_2}\int_{-\infty}^{\infty}\int E(u_1, u_2)\frac{du_1 du_2}{u_1 u_2} \tag{2.3}$$

where $E(u_1, u_2)$ is the characteristic function of the system of random quantities $X(t_1)$ and $X(t_2)$.

For the normal random process one can evaluate the integral (2.2), and the integral (2.3) becomes considerably simplified. Indeed, suppose that x is the mathematical expectation, σ_x^2 the dispersion and $k(\tau)$ the normalized correlation functions of the random process $X(t)$. Then

$$E(u) = \exp\left(-\frac{\sigma_x^2}{2}u^2 + iux\right) \tag{2.4}$$

$$E(u_1, u_2) = \exp\left(-\frac{\sigma_x^2}{2}[u_1^2 + u_2^2 + 2k(\tau)u_1u_2] + iu_1x + iu_2x\right) \tag{2.5}$$

and in place of (2.2) we obtain

$$\mu_1 = M\{f_1[X(t)]\} = \Phi\left(\frac{x}{\sigma_x}\right) \qquad \left(\Phi(z) = \frac{2}{\sqrt{2\pi}}\int_0^z e^{-\frac{x^2}{2}}dx\right) \tag{2.6}$$

In the evaluation of (2.3) we must note that the integral

$$J(\gamma) = \int_{-\infty}^{\infty}\int \exp\{-au_1^2 - bu_2^2 - \gamma u_1u_2 + i(x_1u_1 + x_2u_2)\}\frac{du_1du_2}{u_1u_2} \tag{2.7}$$

can be transformed, through differentiation with respect to γ, into the form

$$J(\gamma) = -2\pi\int_0^{\gamma}\exp\left\{-\frac{ax_2^2 + bx_1^2 - \xi x_1x_2}{4ab - \xi^2}\right\}\frac{d\xi}{\sqrt{4ab - \xi^2}} + J(0) \tag{2.8}$$

where

$$J(0) = -\pi^2\Phi\left(\frac{x_1}{\sqrt{2a}}\right)\Phi\left(\frac{x_2}{\sqrt{2b}}\right) \tag{2.9}$$

Therefore, in view of (2.5) and (2.9), we shall have in place of (2.3) the expression

$$\nu_1(\tau) = \frac{2}{\pi}\int_0^{k(\tau)}\exp\left(-\frac{x^2}{\sigma_x^2(1+\xi)}\right)\frac{d\xi}{\sqrt{1-\xi^2}} + \Phi^2\left(\frac{x}{\sigma_x}\right) \tag{2.10}$$

When $x = 0$, the integral (2.10) can be evaluated by elementary means. This yields

$$\nu_1(\tau) = \frac{2}{\pi}\sin^{-1}k(\tau) \tag{2.11}$$

In an analogous manner, one can evaluate the moments of the nonlinear expression $f_2[X(t)]$, which on the basis of (1.7) and (1.10) can be represented in the form

$$f_2[X(t)] = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \{\exp[iu(X-a)] + \exp[iu(X+a)]\} \frac{du}{u} \tag{2.12}$$

Evaluating the mathematical expectation of both parts of this equation, we obtain

$$\mu_2 = \frac{1}{2}\left\{\frac{1}{\pi i}\int_{-\infty}^{\infty} e^{iau} E(u)\frac{du}{u} + \frac{1}{\pi i}\int_{-\infty}^{\infty} e^{-iau} E(u)\frac{du}{u}\right\}$$

Since the multiplication of a characteristic function by $\exp[\pm iau]$ is equivalent to the addition of $\pm a$ to the mathematical expectation, we obtain on the basis of (2.2) and (2.6) for the normal random process the expression

$$\mu_2 = \frac{1}{2}\left[\Phi\left(\frac{x+a}{\sigma_x}\right) + \Phi\left(\frac{x-a}{\sigma_x}\right)\right] \tag{2.13}$$

Analogously, the substitution of (2.12) in the product $f_2[X(t_1)]$ $f_2[X(t_2)]$ yields

$$\nu_2 = -\frac{1}{2\pi^2}\int\int_{-\infty}^{\infty}\{\exp[ia(u_1+u_2)] + \exp[ia(u_1-u_2)]\}\; E(u_1, u_2)\frac{du_1du_2}{u_1u_2}$$

Making use of the integral (2.8), we obtain

$$\nu_2 = \frac{1}{\pi}\left\{\int_0^{k(\tau)} \frac{d\xi}{\sqrt{1-\xi^2}}\left[\exp\left(-\frac{(x^2+a^2)}{\sigma_x^2(1-\xi)}\right) + \right.\right. \tag{2.14}$$

$$\left.\left. + \exp\left(-\frac{(x^2+a^2)-(x^2-a^2)\xi}{\sigma_x^2(1-\xi^2)}\right)\right] + \pi\left[\Phi\left(\frac{x+a}{\sigma_x}\right) + \Phi\left(\frac{x-a}{\sigma_x}\right)\right]^2 \Phi\left(\frac{x+a}{\sigma_x}\right)\right\}$$

When $x = 0$ the last formula reduces to the simpler form

$$\nu_2 = \frac{1}{\pi}\int_0^{k(\tau)}\left[\exp\left(-\frac{a^2}{\sigma_x^2(1+\xi)}\right) + \exp\left(-\frac{a^2}{\sigma_x^2(1-\xi)}\right)\right]\frac{d\xi}{\sqrt{1-\xi^2}} \tag{2.15}$$

For the nonlinear expression (1.4) we have, in view of (1.10)

$$f_3[X(t)] = \frac{1}{2\pi i}\left\{\int_{-\infty}^{\infty} \exp[iu(X+a)](X+a)\frac{du}{u} - \right.$$

$$\left. - \int_{-\infty}^{\infty} \exp[iu(X-a)](X-a)\frac{du}{u}\right\} \tag{2.16}$$

Evaluating the mathematical expectation of both parts of the last equation, and taking into account the fact that

$$M\{e^{iuX(t)}X(t)\} = \frac{1}{i}\frac{\partial}{\partial u}E(u)$$

we obtain

$$\mu_3 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{a}{i}(e^{iau}+e^{-iau})E(u)-(e^{iau}-e^{-iau})\frac{\partial E(u)}{\partial u}\right\}\frac{du}{u}$$

which for the normal law becomes

$$\mu_3 = \frac{1}{2}\left[(x+a)\Phi\left(\frac{x+a}{\sigma_x}\right)-(x-a)\Phi\left(\frac{x-a}{\sigma_x}\right)\right]+ \\ +\frac{\sigma_x}{\sqrt{2\pi}}\left[\exp\left(-\frac{(x+a)^2}{2\sigma_x{}^2}\right)-\exp\left(-\frac{(x-a)^2}{2\sigma_x{}^2}\right)\right] \qquad (2.17)$$

Making use of (2.16) in the evaluation of $\nu_3(\tau)$, we obtain, in the case of the normal random process with zero mathematical expectation, the following expression:

$$\nu_3 = -\frac{a^2+\sigma^2k}{\pi}\int_0^k\exp\left(-\frac{a^2}{\sigma^2(1+\xi)}\right)\left[1-\frac{2a^2(1+k)}{(a^2+\sigma^2k)(1-\xi)}+\right. \\ \left.+\frac{2k}{a^2+\sigma^2k}\frac{a^2(1-\xi)^2-\sigma^2(1-\xi^2)}{(1-\xi^2)^2}\right]\frac{d\xi}{\sqrt{1-\xi^2}}- \\ -\frac{a^2-\sigma^2k}{\pi}\int_0^k\exp\left(-\frac{a^2}{\sigma^2(1-\xi)}\right)\left[1-\frac{2a^2(1-k)}{(a^2-\sigma^2k)(1+\xi)}-\right. \\ \left.-\frac{2k}{a^2-\sigma^2k}\frac{a^2(1+\xi)^2-\sigma^2(1-\xi^2)}{(1-\xi^2)^2}\right]\frac{d\xi}{\sqrt{1-\xi^2}}+ \\ +\frac{\sigma^2(1+k^2)}{\pi\sqrt{1-k^2}}\left[\exp\left(-\frac{a^2}{\sigma^2(1-k)}\right)-\exp\left(-\frac{a^2}{\sigma^2(1+k)}\right)\right] \qquad (2.18)$$

The derived moments of the nonlinear expressions $f_j[X(t)]$ make it possible to compute the mathematical expectation and the dispersion of the solution of Equation (1.1) which are given by Formulas (1.7) and (1.8).

3. Examples on the application of the method. 1. As the simplest type of example on the application of the method developed in this work, let us consider the equation

$$\frac{d}{dt}Y(t) = m + n\,\mathrm{sign}\,X(t) \qquad (3.1)$$

This equation describes, for example, the deviation $Y(t)$ of the axis of a gyroscope in consequence of Coulomb friction in the supports, if the gyroscope is installed on a ship in such a manner that the horizontal axis of the support coincides with the diametral plane of the ship. In this case the constants m and n are determined by the structural parameters of the gyroscope, while the random function $X(t)$ is the angular velocity of ship's rolling. The angle $\theta(t)$ of the heeling of the ship can be considered to be a stationary normal random function of time whose correlation function is given by the equation

$$K_\theta(\tau) = \sigma_\theta^2 e^{-\alpha|\tau|}\left(\cos\beta\tau + \frac{\alpha}{\beta}\sin\beta|\tau|\right) \tag{3.2}$$

where the constants α, β and σ_θ depend on the nature of the waves of the sea and on the parameters of the ship.

Since the integration of Equation (3.1) for zero initial conditions yields

$$Y(t) = mt + n\int_0^t \operatorname{sign} X(t_1)\,dt_1 \tag{3.3}$$

the application of Formulas (2.6), (2.11), (1.7) and (1.8) will give

$$y(t) = mt, \qquad \sigma_y^2 = 4\pi n^2\int_0^t (t-\tau)\sin^{-1} k(\tau)\,dt \tag{3.4}$$

where, in accordance with (3.2), we have

$$k(\tau) = e^{-\alpha|\tau|}\left(\cos\beta\tau + \frac{\alpha}{\beta}\sin\beta|\tau|\right)$$

When $t \gg 1/\alpha$, the upper limit of integration in the integral (3.4) can be considered as infinite. This yields

$$\sigma_y^2 \approx at - b \qquad \left(a = 4\pi n^2\int_0^\infty \sin^{-1} k(\tau)\,d\tau,\quad b = 4\pi n^2\int_0^\infty \tau\sin^{-1} k(\tau)\,d\tau\right) \tag{3.5}$$

In the given example we shall make use of the method of statistical linearization [2], that is, we shall make use of the approximating substitution

$$\operatorname{sign} X(t) = AX(t) + B \tag{3.6}$$

where the constants A and B are so chosen that the first and second moments on the left- and right-hand sides of (3.6) are equal. In this manner we obtain

$$\sigma_y^2 = 2n^2A^2\sigma_\theta^2[1 - k_\theta(t)] \tag{3.7}$$

Therefore, in the given case, the method of statistical linearization gives even a qualitatively incorrect result: in place of the linear dependence on time we obtain Expression (3.7) which tends to the constant value $2n^2A^2\sigma_0^2$.

2. As a more complicated example let us consider the system of equations

$$\frac{d^2\beta(t)}{dt^2} - p\frac{d}{dt}\alpha(t) = -k_1 \operatorname{sign} X_1(t), \quad \frac{d^2}{dt^2}\alpha(t) + q\frac{d}{dt}\beta(t) = k_2 \operatorname{sign} X_2(t) \qquad (3.8)$$

where p, q, k_1 and k_2 are constants, while $X_1(t)$ and $X_2(t)$ are independent stationary functions of time. Such a type of equation arises, for example, in the investigation of the behavior of a gyrovertical installed on a rolling boat with Coulomb friction between the axes and the support of the gyroscope. In this case, $X_1(t)$ is the angular velocity of heaving, $X_2(t)$ is the angular velocity of rolling, and the coefficients of the equation are determined by the structural parameters of the gyroscope. For the solution of this problem it is convenient to introduce into (3.8) a complex variable

$$\zeta(t) = \frac{1}{\sqrt{q}}\alpha(t) + i\frac{1}{\sqrt{p}}\beta(t) \qquad (3.9)$$

After this, the original system of equations can be replaced by one equation

$$\frac{d^2}{dt^2}\zeta(t) - i\lambda\frac{d}{dt}\zeta(t) = -i\varkappa_1 \operatorname{sign} X_1(t) + \varkappa_2 \operatorname{sign} X_2(t) \qquad (3.10)$$

where

$$\lambda = \sqrt{pq}, \qquad \varkappa_1 = \frac{k_1}{\sqrt{p}}, \qquad \varkappa_2 = \frac{k_2}{\sqrt{q}}$$

The solution of (3.10) under zero initial conditions will be

$$\zeta(t) = \frac{1}{i\lambda}\int_0^t [e^{i\lambda(t-t_1)} - 1][\varkappa_2 \operatorname{sign} X_2(t_1) - i\varkappa_1 \operatorname{sign} X_1(t_1)]\,dt_1 \qquad (3.11)$$

Taking into account (3.9), we can find the mathematical expectation and dispersion of the functions $\alpha(t)$ and $\beta(t)$ by means of the relations

$$M[\alpha(t)] = \frac{1}{2}\sqrt{q}\{M[\zeta(t)] + M[\zeta^*(t)]\}, \qquad M[\beta(t)] = \frac{1}{2i}\sqrt{p}\{M[\zeta(t)] - M[\zeta^*(t)]\} \qquad (3.12)$$

$$\frac{1}{q}D[\alpha(t)] + \frac{1}{p}D[\beta(t)] = M\{|\zeta(t)|^2\} - \frac{1}{q}\{M[\alpha(t)]\}^2 - \frac{1}{p}\{M[\beta(t)]\}^2 \qquad (3.13)$$

$$\frac{1}{q}D[\alpha(t)] - \frac{1}{p}D[\beta(t)] = \frac{1}{2}M\{[\zeta(t)]^2 + [\zeta^*(t)]^2\} - \frac{1}{q}\{M[\alpha(t)]\}^2 + \frac{1}{p}\{M[\beta(t)]\}^2$$

Having made use of the integral representation (1.10), we can express all mathematical expectations which appear in (3.12) and (3.13) in terms of the characteristic function of the ordinates of the random functions $X_1(t)$ and $X_2(t)$. Since these computations are entirely analogous to those of the preceding section, we shall give only the final results which were obtained for the normal random processes $X_1(t)$ and $X_2(t)$ with zero mathematical expectations. In this case

$$M[\zeta(t)] = 0, \quad \text{or} \quad M[\alpha(t)] = 0, \qquad M[\beta(t)] = 0$$

$$M\{|\zeta(t)|^2\} = \frac{4}{\pi\lambda^2}\int_0^t \left\{(t-\tau)(\cos\lambda\tau + 1) + \frac{1}{\lambda}[\sin\lambda\tau - \sin\lambda(t-\tau) - \sin\lambda t]\right\} \times$$
$$\times [\varkappa_2^2 \sin^{-1} k_2(\tau) + \varkappa_1^2 \sin^{-1} k_1(\tau)]\,d\tau \tag{3.14}$$

$$M\{[\zeta(t)]^2\} = -\frac{4}{\pi\lambda^2}\int_0^t \left\{(t-\tau) + \frac{1}{i\lambda}\left[1 + \frac{1}{2}e^{i\lambda\tau} - \frac{1}{2}e^{i\lambda t} + \frac{1}{2}e^{i\lambda(2t-\tau)} - \right.\right.$$
$$\left.\left. - e^{i\lambda(t-\tau)}\right]\right\}[\varkappa_2^2 \sin^{-1} k_2(\tau) - \varkappa_1^2 \sin^{-1} k_1(\tau)]\,d\tau \tag{3.15}$$

where $k_1(\tau)$ and $k_2(\tau)$ are normalized correlation functions $X_1(t)$ and $X_2(t)$, respectively. Formulas (3.14) and (3.15), together with (3.13), permit one to compute the values of the dispersions $D[\alpha(t)]$ and $D[\beta(t)]$.

In an analogous manner one can carry out the solution to the end for other types of nonlinearities that may appear on the right-hand sides of Equations (3.8).

4. The application of the method to more complicated problems. The complication of the problem can occur in three directions. Firstly, one can do away with the normality and the stationary nature of the random functions which appear in the equation in a nonlinear manner. This generalization does not introduce any major difficulties. The difference consists only in the fact that the characteristic functions which appear during the process of evaluating the dispersion of the solutions of the equations will be of greater complexity. Secondly, one may pass to the consideration of nonlinear expressions of the hysterisis type of characteristics, for example, of the form

$$f[X(t)] = \begin{cases} +1, & \text{if } X(t) > a, \text{ or } -a \leqslant X(t) \leqslant a, \text{ but } X(t) \text{ lies in the interval } (-a, a) \text{ intersecting the level } X(t) = a \text{ from above} \\ -1, & \text{if } X(t) < -a, \text{ or } -a \leqslant X(t) \leqslant a, \text{ but } X(t) \text{ lies on the interval } (-a, a) \text{ intersecting the level } X(t) = -a \text{ from below} \end{cases} \tag{4.1}$$

The nonlinearities of this type can be expressed explicitly in terms of the ordinates of the function $X(t)$, but the integral expressions which one obtains hereby turn out to be more complicated than they are for the nonlinearities treated in the present work; the derivation of the mathematical expectation and dispersion of the solution of the equations becomes considerably more involved.

The third direction in which one can advance in order to complicate the problem is the consideration of equations in which the arguments of the nonlinear terms are not random functions with given probability characteristics but are algebraic sums of such functions and of the solution function. An example of this type is furnished by the equation

$$\frac{d}{dt} Y(t) = m + nf_j [X(t) - Y(t)] \tag{4.2}$$

or by the equation

$$\frac{d^n Y(t)}{dt^n} + a_1(t) \frac{d^{n-1} Y(t)}{dt^{n-1}} + \cdots + a_n(t) Y(t) = f_j [X(t) - Y(t)] \tag{4.3}$$

which differ from Equations (3.1) and (1.1), respectively, only in their right-hand sides.

It is not possible to obtain a general method of solution for these problems as easily as for problems in which the arguments of the nonlinear functions are given random functions. Nevertheless, the method suggested in the present work is useful for obtaining an approximate solution in case the solution of the equation under consideration may be assumed to be small compared to the ordinates of the disturbances of the random function. In such a case one can find the solution of an equation of the type (4.3) by the method of successive approximations, whose essence consists of the following. First, one drops the solution function which appears in the argument of the nonlinearity, and one determines the moments of the solutions of the obtained equations in the same way as was done above for the mathematical expectation and dispersion. After this one repeats the calculation from the beginning, but now one replaces the characteristic functions of the given random functions by the characteristic functions of the algebraic sums of these functions and of the solutions of the equations found in the first approximation. Since the characteristic functions will not be normalized, one may use for their evaluation Edgeworth's series [4], keeping in it only those terms which can have a noticeable effect on the result. The moments of the second approximate solutions found in this manner can be used for the derivation of the succeeding approximations. The finding of each successive approximation is connected with great difficulties. Therefore, in problems of the indicated type, the given method is to be recommended basically only then when the error in the first approximation is sought, and when it can be assumed from general considerations that the first approximation will give satisfactory results.

We shall explain what we have said with the example of Equation (4.2), which for $j = 1$ describes the deviation of a gyrovertical, with a contact characteristic correction, installed on a rolling ship. In this case the normal random function $X(t)$, which describes the random deflection of the pendulum of the deflector, has a dispersion which is considerably greater than the dispersion of the angle $Y(t)$ characterizing the deviation of the gyrovertical. Therefore, the application of the method of successive approximations is expedient. Dropping in the first approximation $Y(t)$ on the right-hand side of (4.2), we obtain Equation (3.1). The dispersion of its solution is given by (3.6). For the purpose of finding the correlation function $Y(t)$ in its first approximation, which is required for finding the dispersion $Y(t)$ in the second approximation, it is sufficient to multiply (3.4) term-wise by a similar equation written for another argument t, and with the use of (1.10) to find the mathematical expectation of each part of the obtained equation. After some transformations analogous to those used in Sections 2 and 3, we obtain the next expression for the correlation function of $Y(t)$:

$$K_y(t_1, t_2) = \frac{2n^2}{\pi} \int_0^{t_2}\int_0^{t_1} \sin^{-1} k(\xi - \eta)\, d\xi\, d\eta \tag{4.4}$$

Analogous arguments yield for the first approximation of the correlation function of the connection $R_{xy}(t_1, t_2)$ of the random functions $X(t)$ and $Y(t)$

$$R_{xy}(t_1, t_2) = n\sqrt{\frac{2}{\pi}} \int_0^{t_2} k(t_1 - \xi)\, d\xi \tag{4.5}$$

For finding the third, fourth and higher moments of $Y(t)$ it is necessary to take the product of three, four and more expressions of the type (3.4) for various values of the argument t, and to determine the mathematical expectations of both parts of the obtained equations. The order of computation remains the same but the number of integrations is increased, which complicates the process of computation. In order to find the second approximations of the moments of $Y(t)$ one begins with the equation

$$Y_2(t) = mt + \frac{n}{\pi i} \int_0^{t}\int_{-\infty}^{\infty} \exp\{iu\,[X(\xi) - Y_1(\xi)]\}\, \frac{du}{u}\, d\xi$$

where the indices 1 and 2 denote the first and second approximations. Since the dispersion of $Y_1(t)$ is small compared to the dispersion of $X(t)$, the difference

$$Z(\xi) = X(\xi) - Y_1(\xi)$$

can be considered a normal random function. Hence, repeating the calculations of Section 2 for the dispersion of $Y_2(t)$, we obtain

$$D[Y_2(t)] = \frac{2\pi^2}{\pi} \int_0^{t}\int_0^{t} \sin^{-1} k_z(\xi, \eta)\, d\xi\, d\eta$$

where $k_2(t_1, t_2)$ is the normalized correlation function of $Z(t)$ determined by the equation

$$k_z(t_1, t_2) = \frac{\sigma_x^2 + K_y(t_1, t_2) - R_{xy}(t_1, t_2) - R_{yx}(t_1, t_2)}{\sqrt{\{\sigma_x^2 + D[Y_1(t_1)] - 2R_{yx}(t_1, t_1)\}\{\sigma_x^2 + D[Y_1(t_2)] - 2R_{yx}(t_2, t_2)\}}}$$

where $K_y(t_1, t_2)$ and $R_{xy}(t_1, t_2)$ are taken from the first approximation.

The succeeding approximations can be obtained in a similar way.

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